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RATIONAL CALCULATION OF LAMINAR AND TURBULENT COMPRESSIBLE BOUNDARY LAYERS WITH HEAT TRANSFER[1]

N. Scholz

ABSTRACT: An approximate procedure for the calculation of laminar and turbulent boundary layers with available pressure and wall temperature is given for engineering use. The influence of compressibility and heat transfer on the boundary layer flow is taken into account. Some results of calculation are communicated.

1. Statement of the Problem /33*

The following considerations concerning the calculation of compressible boundary layers are intended primarily as a contribution to the practical aspects of boundary layer theory. The success of boundary layer theory in the treatment of engineering problems and especially in the development of schematic methods of calculation, capable of application by nonspecialized individuals, is making them an ever increasing and valuable aid in industrial development. The problems encountered in this work are mainly of a highly complex nature, so that a more or less idealized statement of the problem must be formulated for a theoretical treatment. It is precisely for this range of application that calculation methods have been employed which are capable of describing the important content of the process while disregarding certain physical details and whose treatment is as simple and universal as possible. These preliminary remarks are made because the method of calculation proposed below will admittedly fall short of providing complete satisfaction in the theoretical respect, but provide the developmental engineer with a tool that he can use to obtain a rapid (albeit frequently quite rough) answer to the numerous problems encountered in practice.

[1] Delivered on October 10, 1958 at the annual meeting of the WGL in Stuttgart.

*Numbers in the margin indicate pagination in the foreign text.

As is known, the flow of a medium along a thick boundary, in addition to the flow boundary layer in which the velocity decreases from the total value of the external flow to a zero value at the wall, in the general case also involves a temperature boundary layer in which the static temperature of the flow medium changes from the external flow value to the wall temperature.

Hence, the deviations in temperature within the boundary layer may be caused either by a heating of the flow medium as a result of internal friction or by a change in wall temperature caused by the wall itself. The former becomes more significant with increasing Mach number, while the latter occurs when the wall is heated or cooled. In each case, the reactive effect of this temperature boundary layer on the flow boundary layer rests on the dependence of the material properties of the flow medium on the local temperature.

For constant material properties, we already have very satisfactory boundary layer calculation methods which required (in both the laminar and turbulent cases) only the quadrature of certain functions that must be determined from the given friction-free flow. An attempt is made to derive an analogous method that approximately considers also the influence of the variable material properties as a result of frictional heat and heat transfer and leads back to the known quadratures in the boundary case of constant material properties. Hence, it shall be limited to the plane case, mentioning only that the rotation-symmetrical case, as is otherwise conventional, may be included.

2. Symbols /34

x, y	coordinates in the direction of the wall and perpendicular to it
l	reference length, generally the running length of the boundary layer
δ, δ_T	the thickness of the flow and temperature boundary layers
$\bar{\delta}$, $\bar{\delta}_T$	energy loss and heat gain thickness according to equations (7) and (8)
φ, δ^*	momentum loss thickness and compression of flow boundary layer
u, U	velocity in the boundary layer and at the outer edge

T, T_δ	temperature of the flow medium in the boundary layer and at the outer edge
T_W, T_r	wall temperature, forced or in a heat-insulated walls
T_0	isentropic ram temperature
ρ	density of the flow medium outside the boundary layer
$\mu\delta$	viscosity of the flow medium outside the boundary layer
c_p	specific heat of the flow medium outside the boundary layer
λ_W	thermal conductivity of the flow medium at the wall
d	dissipation of the boundary layer profile
g	acceleration due to gravity
Re_∞, $Re_{\bar{\delta}}$	Reynolds numbers $(= \rho_\infty U_\infty l/\mu_\infty, = \rho_\delta U\bar{\delta}/\mu_\delta)$
M	local Mach number of flow ouside the boundary layer
Pr	Prandtl number [for gases $\neq f(T)$]
H	boundary layer shape factor of displacement thickness $(= \delta^*/\varphi)$
$\bar{H}$	boundary layer shape factor of energy loss thickness $(= \bar{\delta}/\varphi)$
$\bar{H}_T$	boundary layer shape factor of heat gain thickness $(= \bar{\delta}_T/\bar{\delta})$
c_f, c_{f0}	coefficient of friction of a flat plate at zero incidence, random or at $M = 0$ and a heat-insulated wall
c_f', c_{f0}'	local coefficient of friction of the flat plate at zero incidence, random or at $M = 0$ and a heat-insulated wall
κ	isentropic exponent
n, $\bar{n}$	exponent of the dissipation statement (10) or the friction law (12)
ω	exponent of the viscosity law (2)
a, b; α, β	constants

Meaning of Subscripts

∞	for incident flow
B	for the reference temperature T_B according to equation (3)
0	for the flat plate at zero incidence with $M = 0$ and a heat-insulated wall (exception for T_0 see above)
$'$	for values that vary with the integration variable x'

3. Reference Temperature of Boundary Layer With Non-Constant Material Properties

In the following, we shall discuss that the flow boundary layer of a gaseous flow medium whose Prandtl number is known to be on the order of 1 and is practically independent of temperature. For this case, the thickness of the temperature boundary layer has approximately the same value as that of the flow boundary layer. A certain change in the density and viscosity develops for the flow boundary layer in addition to the velocity profile due to the changing temperatures inside the boundary layer (Figure 1). With the usual simplification p = const within a boundary layer profile, we obtain the following for the density according to the general gas law

$$\varrho(y)/\varrho_\delta = [T(y)/T_\delta]^{-1} \tag{1}$$

and for the viscosity with the known power law

$$\mu(y)/\mu_\delta = [T(y)/T_\delta]^{\omega}. \tag{2}$$

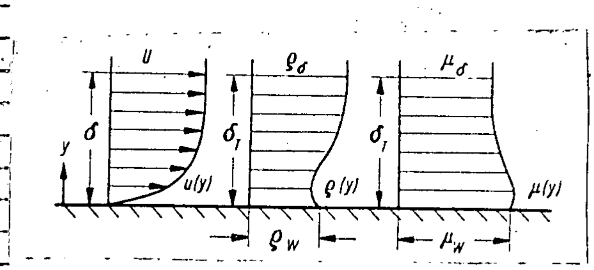

Figure 1. Flow Boundary Layer of a Gaseous Flow Medium (Schematic). Curves of velocity u(y), density ρ(y) and viscosity μ(y) for temperature profile roughly corresponding to that in Figure 2.

We can think of the temperature boundary layer as being composed of the superimposition of two fundamental cases, first the case without heat transfer (a wall not transparent to heat), in which the temperature increases with proximity to the walls due to internal friction and reaches the value T_r at the wall, and secondly the case of a heat transfer at the wall, in which the layer of the flow medium near the wall is influenced by the thermal conduction by means of the induction of a certain wall temperature T_W (heating or cooling) (Figure 2). The gradient dT/dy at the wall determines the direction of the heat transfer.

It is now evident (and has already been proposed by several researchers) [1, 2] that the influence of the different temperatures within the frictional layer can be taken into account by the fact that the material properties /35 of the flow medium can be assumed constant within the frictional layer, but their values will differ from those in the external flow in the sense that they correspond to a certain average temperature within the frictional layer. It shall be designated that this model boundary layer is a quasiconstant set of material properties and the temperature at which the material properties of the boundary layer flow are to be determined will be called the reference temperature T_B. This reference temperature must be composed of the external temperature T_δ and a component that depends on the temperature differential $T_r - T_\delta$ as a result of frictional heat and the temperature differential $T_r - T_W$ due to heat transfer, making up the parts of the temperature boundary layer. An appropriate simple relationship is the following

$$T_B = T_\delta + a(T_r - T_\delta) - b(T_r - T_W) \tag{3}$$

with a and b still unknown dimensionless numbers. Strictly speaking, these numbers will depend on the critical similarity values, namely

a and b equal to functions of P_r, M, Re, H. (4)

In the case of the dependence of the reference temperature T_B on the Prandtl number, experimental studies are available only for heat transfer [3], which show no significant dependence in the area of Prandtl numbers in the vicinity of 1 (gaseous flow medium), so that T_B may be considered as practically independent of the Prandtl number due to the analogy between heat transfer and friction. Dependence on the Mach number was investigated for the laminar boundary layer at constant external velocity by E. Eckert [4] by comparison with a stricter theory [4], where there was practically no dependence in the range $0 < M < 10$. For the case of turbulent boundary layers with constant external velocity the relationship through comparison with the theory of A. Walz [6] was investigated. The result is shown in Figure 3 and likewise shows practically no dependence on the Mach number[1]. Since the form of the boundary layer profile in the case of the laminar boundary layer flow is strictly (in a turbulent boundary layer, with good approximation) independent of the Reynolds number, a and b can also be expected to be independent of the Reynolds number. For the dependence of the boundary layer shape factor H and hence the curve of external flow U(x), due to the lack of suitable comparative values especially for the turbulent boundary layer, little can be said at the present time. Since we obtain a and b from the comparison with results of a boundary layer with constant external velocity, we can disregard the influence of the change in the shape parameter H with respect to the value of the boundary layer of the plate. A similar procedure in the law of energy dissipation of a boundary layer profile led to highly useful results (cf. [7], p. 537), so that the disregard of the influence of H on the reference temperature for an approximate method also appears justified to a certain extent. Deviations can be expected primarily in the case of a sharp rise in pressure (vicinity of the point of shedding).

(4'')

The frictional temperature T_r is known to be a function of the heating factor r and the Mach number of the external flow:

$$T_r = T_\delta\left(1 + r\frac{\varkappa - 1}{2}M^2\right), \tag{5}$$

and in the case of gases with Prandtl number of approximately 1 the function

[1] The equations used to determine a and b from the coefficient of friction of the flat plate are as follows:

$$a = \frac{\left(\frac{c_f}{c_{f_0}}\right)_{T_r}^{-\frac{1+n}{1-n\omega}} - 1}{r\frac{\varkappa-1}{2}M_\infty^2}, \quad b = \frac{\left(\frac{c_f}{c_{f_0}}\right)_{T_r}^{-\frac{1+n}{1-n\omega}} - \left(\frac{c_f}{c_{f_0}}\right)_{T_W}^{-\frac{1+n}{1-n\omega}}}{\frac{T_r - T_W}{T_\infty}}$$

The subscripts T_r and T_W represent the ratio of the coefficients of friction and $T_W = T_r$ and $T_W = \text{const} \neq T_r$. The corresponding formulae with the local coefficients of friction are:

$$a = \frac{\left(\frac{c_f'}{c_{f_0}'}\right)_{T_r}^{-\frac{1}{1-n\omega}} - 1}{r\frac{\varkappa-1}{2}M_\infty^2}, \quad b = \frac{\left(\frac{c_f'}{c_{f_0}'}\right)_{T_r}^{-\frac{1}{1-n\omega}} - \left(\frac{c_f'}{c_{f_0}'}\right)_{T_W}^{-\frac{1}{1-n\omega}}}{\frac{T_r - T_W}{T_\infty}}$$

$r(Pr)$ can be given in good approximation by fractional powers of the Prandtl number (cf. [4]). Finally, we obtain for the reference temperature

$$T_B = T_\delta\left[1 + r\frac{\varkappa - 1}{2} M^2 (a - b) - b\right] + b\,T_W \qquad (6)$$

with the constants according to Table 1.

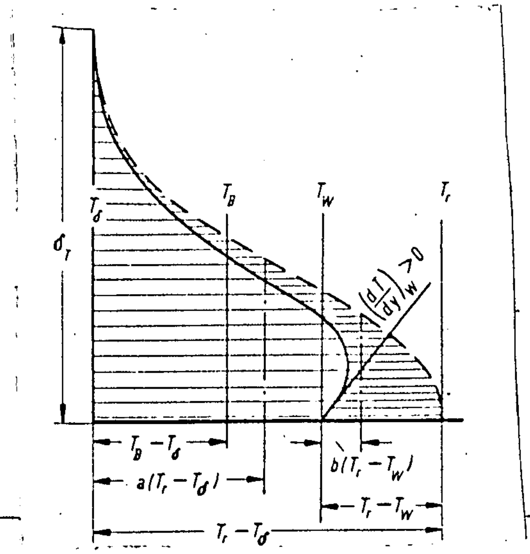

Figure 2. Temperature Boundary Layer of a Flow With Heat Transfer (Schematic). Dashed curve without heat transfer, finely shaded area: temperature component due to heat transfer to the wall. Heat flow from the heat medium → → wall: $(dT/dy)_W > 0$, wall → flow medium: $(dt/dy)_W < 0$.

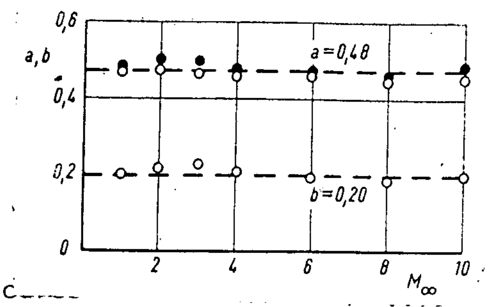

Figure 3. Values a and b of the Reference Temperature According to Equation (3) as a Function of Mach Number of Incident Flow of the Turbulent Plate Boundary Layer
○ obtained from $c_f'(M_\infty, T_r, T_W)$,
● obtained from $c_f(M_\infty, T_r)$, both according to [6].

TABLE 1. VALUES OF THE CONSTANTS FOR THE REFERENCE TEMPERATURE IN EQUATIONS (3) to (6), LAMINAR ACCORDING TO [4], TURBULENT ACCORDING TO FIGURE 3

Frictional layer	a	b	r(Pr)
Laminar	0.72	0.50	$Pr^{1/2}$
Turbulent	0.48	0.20	$Pr^{1/3}$

4. Integration of the Law of Conservation of Energy of the Boundary Layer

For the proposed method of approximation, we shall use the energy law of the boundary layer theory to calculate the energy loss thickness which serves as a measuremof the loss suffered upstream in the boundary layer with respect to kinetic energy of the flow as a result of friction and is defined by the equation

$$\varrho_\delta U^3 \bar{\delta} = \int_0^\delta \varrho(y)\, u(y)\, [U^2 - u^2(y)]\, dy\,. \tag{7}$$

In addition, we shall use a characteristic value of the corresponding temperature boundary layer as the heat-gain thickness, which is a measure of the increase in flow of perceptible heat due to upstream friction and heat gain or loss by the wall and is defined by the equation

$$\varrho_\delta U T_\delta \bar{\delta}_T = \int_0^{\delta_T} \varrho(y)\, u(y)\, [T_{(y)} - T_\delta]\, dy \tag{8}$$ /36

(cf. [7], page 328, where the enthalpies are used instead of the temperature). ~~Finally, the dissipation d disappears in the energy law, representing the~~ (4'') amount of energy converted locally in a boundary layer profile from the principal flow into frictional heat and turbulence (cf. [7], page 537]. The energy law is written with these three values (cf. [7], page 328 ff., but where the limitation of a wall impermeable to heat is made, also [8])

$$\frac{1}{\varrho_\delta U^3}\frac{d(\varrho_\delta U^3 \bar{\delta})}{dx} + 2\bar{\delta}_T \frac{1}{U}\frac{dU}{dx} = 2\frac{d}{\varrho_\delta U^3}. \tag{9}$$

For integration of this equation we require a relationship between the energy loss thickness $\bar{\delta}$ and the value d and $\bar{\delta}_T$. We will obtain the former from a generalizaiton of the corresponding loss for the dissipation in a boundary layer flow with constant material properties according to E. Truckenbrodt [8], which derived from measurements by J. Rotta [9]. Under the assumption used in the previous segment of a boundary layer with quasiconstant material properties, which can be obtained at the reference temperature T_B, this law will read as follows

$$\frac{d}{\varrho_B U^3} = \overline{H}^n \beta \left(\frac{\mu_B}{\varrho_B U \overline{\delta}_B}\right)^n. \tag{10}$$

In addition, the representation of a constant reference temperature T_B within a boundary layer profile will resemble the relationship

$$\varrho_B U^3 \overline{\delta}_B = \varrho_B U^3 \overline{\delta}$$

since in the definition equation (7) for $\overline{\delta}$ the density is treated as constant. The same is true for the momentum loss thickness φ (cf. [7], page 328), from which a boundary layer thickness ratio $\overline{H}$ independent of the temperature distribution is obtained[2]. If we now use (1) and (2) with $\rho(y) = \rho_B$ to introduce the material values of the external flow into (10), we will obtain for the dimensionless dissipation

$$\frac{d}{\varrho_\delta U^3} = \overline{H}^n \beta \left(\frac{\mu_\delta}{\varrho_\delta U \overline{\delta}}\right)^n \left(\frac{T_\delta}{T_B}\right)^{1-n\omega}. \tag{11}$$

This law is given as a working hypothesis, which allows integration of the energy law. Here T_B is selected so that the law is fulfilled as well as possible on the average, proceeding from the consideration in the previous section. The exponent n in (11) can easily be made to agree with the exponent $\overline{n}$ of the friction law of the flat plate with zero incidence with constant material values

$$c_{f_0} = \frac{\alpha}{Re_\infty^{\overline{n}}} \tag{12}$$

while with U = const and the dissipation law (10) the energy equation (9) is integrated. A comparison with (12) then gives us (cf. also [8])

$$n = \frac{\overline{n}}{1-\overline{n}}. \tag{13}$$

Table 2 gives a compilation of the exponent, while for turbulent boundary layers the Prandtl power law of plate friction (cf. [7], page 500) is given

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[2]On the other hand, this is not true for the boundary layer thickness ratio $H = \delta^*/\varphi$, for which the author gives an estimate in [10].

TABLE 2. EXPONENTS OF THE RESISTANCE LAW (12) AND CORRESPONDING EXPONENTS OF THE DISSIPATION LAW (12)

Exponent	$\bar{n}$	n
Laminar	1	1/2
Turbulent	1/5	1/4

A relationship between the energy loss density $\bar{\delta}$ and the heat-gain thickness $\bar{\delta}_T$ can be established by establishing an energy balance between two boundary layer profiles separated in the flow direction. The loss of kinetic energy of the main flow is therefore equated to the increase in heat transferred to the wall by thermal energy created by dissipation and by heat transfer[3].

The following boundary layer values are obtained:

$$\bar{H}_T = \frac{\bar{\delta}_T}{\bar{\delta}} = \frac{\varkappa - 1}{2} M^2 + \frac{T_W - T_\delta}{T_\delta}\left(\frac{T_W}{T_\delta}\right)^{\omega} \frac{1}{Re_{\bar{\delta}} Pr} \int_0^1 Nu(x')\, d\left(\frac{x'}{x}\right) \tag{14}$$

This strictly valid relationship for $\bar{H}_T$ involves a link between the flow and temperature boundary layers, in case there is a heat transfer at the wall

[3]The thermal balance for two points dx′ apart in the boundary layer will be

$$\frac{d}{dx'} \int_0^{\delta} \varrho_\delta u (U^2 - u^2)\, dy = 2 g c_p \frac{d}{dx'} \int_0^{\delta_T} \varrho_\delta u (T - T_\delta)\, dy + 2 \lambda_W \left(\frac{dT}{dy}\right)_W.$$

For the heat flux at the wall, the local Nusselt number is introduced;

$$-\left(\frac{dT}{dy}\right)_W = Nu(x') \frac{T_W - T_\delta}{x}.$$

The integrals are replaced by $\bar{\delta}$ and $\bar{\delta}_T$ according to (7) and (8). Then an integration from 0 to x with x′ as the integration variable following introduction of the speed of sound of the outer flow as well as the Prandtl and Reynolds numbers gives us the equation (14).

(Nu $\neq$ 0). The integral of the second term represents the local Nusselt number obtained upstream from point x, which can be estimated in simple cases. Since the value of $\overline{H}_T$ fortunately is not very much involved in the integration of the energy law, we can content ourselves with estimates of this term. A simple and useful approximation formula[4] is

$$\overline{H}_T = \frac{\varkappa - 1}{2} M^2 + \frac{T_W - T_r}{T_\delta}, \tag{15}$$

where we use average values over the running length of the boundary layer for M and T_W, in order to obtain a value of $\overline{H}_T$ which is independent of x. With the dissipation law (11) and the introduction of $\overline{H}_T$ as an approximately constant average value along the boundary layer, the integration of the energy law (9) may be undertaken. In the following, we shall indicate only a few intermediate steps in this calculation process. The integration takes place within the limits $0 < x' < x$. As a substitution we can use

$$\chi(x') = U^{2\overline{H}_T(1+n)} (\varrho_\delta U^3 \overline{H}\,\overline{\delta})^{1+n} \tag{16}$$

with the derivation

$$\left\{ \begin{aligned} \frac{d\chi}{dx'} &= (1+n)\, U^{2\overline{H}_T(1+n)} (\varrho_\delta U^3 \overline{H}\,\overline{\delta})^n \times \\ &\quad \times \left[\frac{d(\varrho_\delta U^3 \overline{H}\,\overline{\delta})}{dx'} + 2\overline{H}_T \frac{\varrho_\delta U^3 \overline{H}\,\overline{\delta}}{U} \frac{dU}{dx'}\right]. \end{aligned} \right. \tag{17}$$

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[4]This is obtained from the related assumption that the temperature profile of the boundary layer is similar to the profile of the velocity quadrate of the of the flow boundary layer, i.e.;

$$\frac{T_W - T(y)}{T_W - T_\delta} = \frac{u^2(y)}{U^2}$$

Inasmuch as $\delta_T \approx \delta$, we must assume that Pr ≈ 1. By introducing the relationship into the definition equations (7) and (8), we will obtain

$$\overline{H}_T = \frac{T_r - T_\delta}{T_\delta} + \frac{T_W - T_r}{T_\delta} = r\frac{\varkappa - 1}{2} M^2 + \frac{T_W - T_r}{T_\delta}$$

The first term for r - 1 (Pr - 1) agrees with the exact result (14). We shall correct the result by replacing the factor of r by 1 and thus obtain (15), which offers useful values for Prandtl numbers in the vicinity of 1, and for temperature differentials that are not too great.

Using (16), we will obtain the following from the energy law (9)

$$\left\{\begin{aligned} &\frac{d}{dx'}(\varrho_\delta U^3 \overline{H}\,\overline{\delta}) + 2\overline{H}_T \frac{\varrho_\delta U^3 \overline{H}\,\overline{\delta}}{U}\frac{dU}{dx'} = \\ &\qquad = 2\beta\left(\frac{T_\delta}{T_B}\right)^{1-n\omega}\left(\frac{\mu_\delta}{\varrho_\delta U \overline{\delta}}\right)^n \varrho_\delta U^3, \end{aligned}\right. \tag{18}$$

where the left side shows the brackets in (17). Substitution of (18) into (17) allows separation of the variables, since $\overline{\delta}$ drops out. Then we will have

$$\left\{\begin{aligned} &U^{2\overline{H}_T(1+n)}(\varrho_\delta U^3 \overline{\delta})^{1+n} = \\ &\qquad = \int_0^x 2(1+n)\beta \overline{H}^n \varrho_\delta' \mu_\delta'^n U'^{3+2n+2\overline{H}_T(1+n)} \times \\ &\qquad\qquad \times \left(\frac{T'}{T_B'}\right)^{1-n\omega} dx'. \end{aligned}\right. \tag{19}$$

As E. Truckenbrodt [8] has shown,

$$2(1+n)\beta\overline{H}^n = E \tag{20}$$

(4'')

is a function that is largely independent of the shape factor H and hence the running length x, so that it can be extracted before the integral. The solution of (19) for the desired value of $\overline{\delta}$, being made dimensionless with the values of the incident flow (subscript ∞) and the boundary layer length 1, finally gives us the equation

$$\frac{\overline{\delta}}{l} = \frac{\overline{H}_0 c_{f0}}{2}\,\frac{\left[\int_x^0 \left(\frac{T'_\omega}{T'_B}\right)^{1-n\omega}\left(\frac{U'}{U_\infty}\right)^{3+2n+2\overline{H}_T(1+n)} d\left(\frac{x'}{l}\right)\right]^{\frac{1}{1+n}}}{\frac{\varrho_\delta}{\varrho_\infty}\left(\frac{U}{U_\infty}\right)^{3+2\overline{H}_T}} \tag{21}$$

The first factor is obtained from a comparison with the flat plate boundary layer with zero incidence at M = 0. c_{f0} is the incompressible coefficient of friction of the plate with zero incidence for the Reynolds number H_0 of the corresponding formed parameter formed with the values of the incident flow, which in the laminar case has a value of 1.56 and in the turbulent case with $Re = 10^7$ is approximately 1.80 (cf. [10]). For $T_B = T_\infty$, $\rho_\delta = \rho_\infty$ and $\overline{H}_T = 0$, (21) becomes an equation for the boundary layer without frictional heat and

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heat transfer (cf. [8] and [10]). It is valid both for laminar and turbulent flow and requires for its evaluation only a quadrature over a function of the given velocity distribution along the wall.

To determine the momentum loss thickness as well as the sensitivity to solution, a determination of the local shape factor $\bar{H}(x)$ is necessary, but we shall not discuss it here. In reference to the quadrature method of Truckenbrodt [8], this comparatively simple method can be carried out, as we have discussed elsewhere [10].

5. Sample Applications

For the case of the flat plate with zero incidence, we obtain from (21) with

$$\frac{c_f}{2} = \frac{\bar{H}_0 \bar{\delta}}{l}$$

the ratio of the coefficient of friction of the compressible boundary layer to the incompressible boundary layer

(4'')

$$\frac{c_f}{c_{f0}} = \left(\frac{T_\infty}{T_B}\right)^{\frac{1-n\omega}{1+n}}. \qquad (22)$$

For the case of air as a flow medium, we have evaluated this equation in Figure 4 for a wall which is impermeable to heat as a function of Mach number. In accordance to the determination of the constants for the reference temperature (3), the curves in the laminar case agree with the theory of Young and Janssen [5], in the turbulent case with Re = 10^7 with the theory of Walz [6]. The curve with the Reynolds number in the turbulent case comes from the change in the exponent n, which it is related by (13) to the exponent $\bar{n}$ of the resistance law of the flat plate with zero incidence. On the basis of the turbulent Prandtl-Schlichting resistance law ([7], page 502), the latter is obtained from the slope of the resistance curve and is equal to

$$\bar{n} = \frac{1{,}12}{\log Re_l}. \qquad (23)$$

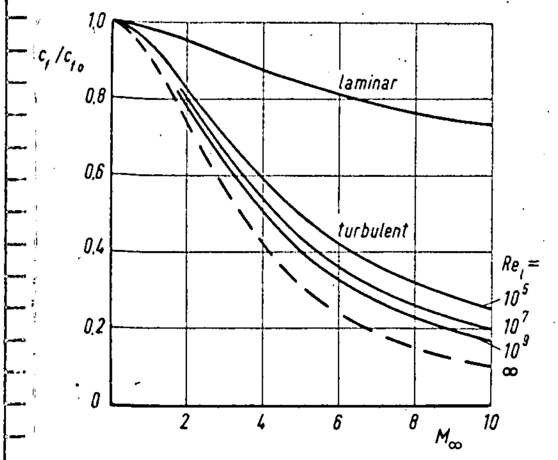

Figure 4. Ratio of the Coefficient of Friction of Incompressible Boundary Layer on a Flat Plate with Zero Incidence Without Heat Transfer as a Function of the Mach Number of the Incident Flow. Flow medium: air with Pr = 0.72, ω = 0.76 and n = 1 (laminar) and n according to equations (13) and (23) (turbulent). ---, Boundary case $Re_\infty \to \infty$ with $n \to 0$.

Figure 5 shows the ratio of coefficients of friction for the flat plate with zero incidence with heat transfer. The abscissa is a ratio of the wall temperature T_W which is constant along the plate to the isentropic ram temperature T_0 of the incident flow. The boundary between the cooling and heating of the wall is given by $T_W = T_r$. The friction is increased by cooling the wall and reduced by heating it.

In Figure 6, for the case of air, we have the boundary (4") layer in a lenticular profile with 5% relative thickness calculated for an incident flow (Mach number M_∞ = 3). In order to show the characteristic differences in the boundary layer on the flat plate, we have based the energy loss thickness of the boundary layer along the profile contour in each case on the local energy loss thickness of the incompressible boundary layer on the plate. The temperature is equal on the one hand to the frictional temperature (no heat transfer, dashed lines) and secondly equal to 400°K at an external temperature of 216.5°K, corresponding to an altitude in the atmosphere of 11 km. (heat transfer from the air to the wall, dashed lines); the horizontal lines give the decrease in frictional losses of the compressible boundary layer at the flat plate. The energy loss thickness is initially less than that at the plate and rises above the plate value toward the rear edge of the profile. This tendency is evoked primarily by the density of the external flow at the lentincular profile which decreases toward the rear edge (the Mach number increases linearly with x). If the wall cooling is limited to the rear half of the profile, the curve following the dashed path will deviate from the solid curve for the front half. If only the front half is cooled, the solid curve that is a continuation of the dashed curve will result in the rear half. The cooling in the front half increases the energy loss thickness more intensely than the cooling of the rear half. This is interesting, since the heat transfer in the front part of the profile is better than in the rear.

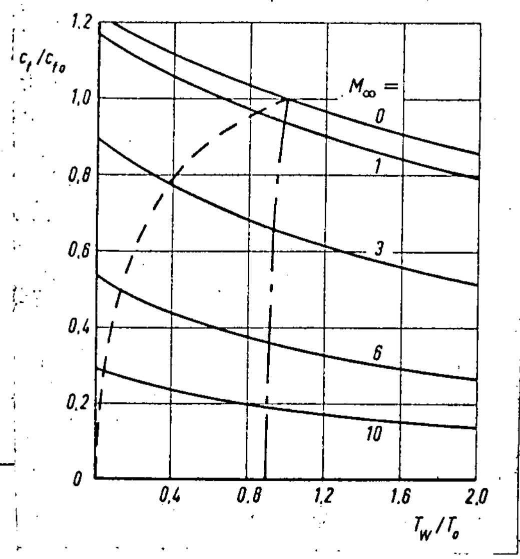

Figure 5. Ratio of the Coefficients of Friction of Compressible to Incompressible Boundary Layers on a Flat Plate with Zero Incidence with Heat Transfer as a Function of Wall Temperature. Flow medium: air with Pr = 0.72 and ω = 0.76, turbulent, $Re_\infty = 10^7$ with n = 0.19.
$-\cdot-$, $T_W = T_r$, no heat transfer;
$-$ $T_W = T_\infty$, cooling of the wall to external temperature.

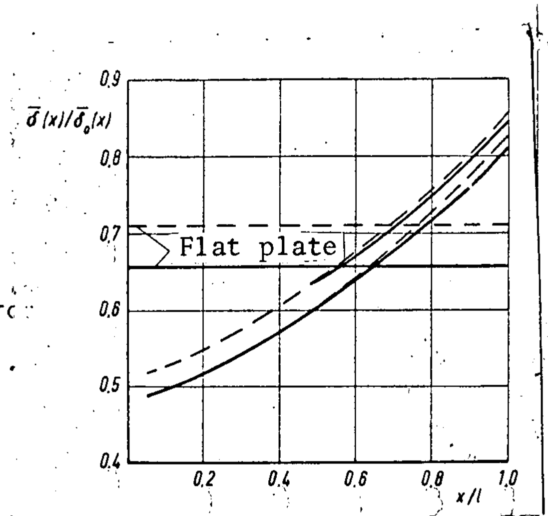

Figure 6. Curve of the Energy Loss Density Relative to a 5% Thick Lenticular Profile with M_∞ = 3 Along the Depth of the Profile. ——, $T_W = T_r$, ---, T_W = 400°K at T_∞ = 216.5°K.

The reference value $\delta_0(x)$ is the local energy loss density of the boundary layer on the flat plate with incompressible flow.

Flow medium: air with Pr = 0.72, ω = 0.76 and $Re_\infty = 10^7$, turbulent.

The frictional resistance of the lenticular profile is obtained from the energy loss thickness $\bar{\delta}_H$ at the rear edge of the profile. If we relate the coefficient of friction again to that of the flat plate without heat transfer at M = 0, we will have

$$\frac{(c_f)_{\text{Profile}}}{c_{f0}} = \frac{c_f}{c_{f0}} \frac{\bar{\delta}_H}{(\bar{\delta}_H)_0} \frac{\varrho_H}{\varrho_\infty} \left(\frac{M_H}{M_\infty}\right)^3 . \qquad (24)$$

The values of this ratio between the coefficients of friction have been summarized in Table 3 for the different assumptions of wall temperature. The second column gives the percentile deviation from the value of the flat plate impervious to heat for a given Mach number. Throughout there is a slight decrease in the frictional resistance of the lenticular profile with respect to the plate with zero incidence with equal wall temperature, caused by the pressure drop along the wall contour of the lenticular profile. A. D. Young and S. Kirkby [11], for a 5% thick lenticular profile with heat-impervious wall, with $M_\infty = 3$ (interpolated) and $Re_\infty = 10^7$, calculate the value

$$(c_f)_{\text{Profile}}/c_{f0} = 0.630$$

in contrast to our value of 0.639. For the total resistance, we add to this the pressure resistance.

TABLE 3. VALUES OF RATIO OF COEFFICIENTS OF FRICTION FOR A FLAT PLATE AND 5% LENTICULAR PROFILE IN AN INCIDENT FLOW OF $M_\infty = 3$) FLOW MEDIUM: AIR WITH Pr = 0.72, ω = 0.76, TURBULENT BOUNDARY LAYER WITH $R_\infty = 10^7$).

Contour	Wall temperature	c_f/c_{f_0}	$\Delta(c_f)\ T_W = T_r$
Flat plate	$0 < x < l: T_W = T_r$	0,658	—
	$0 < x < l: T_W = 400°$	0,710	+ 8,0 %
Lenticular profile, d/l = 5%	$0 < x < l: T_W = T_r$	0,639	− 2,9 %
	$0 < x < l: T_W = 400°$	0,675	+ 2,6 %
	$0 < x < l/2: T_W = T_r$ $l/2 < x < l: T_W = 400°$	0,650	− 1,2 %
	$0 < x < l/2: T_W = T_r$ $l/2 < x < l: T_W = 400°$	0,665	+ 1,1 %

Note: commas indicate decimal points.

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